COUPLED FIXED POINT THEOREMS FOR ϕ-CONTRACTIVE MIXED MONOTONE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper we extend the coupled fixed point theorems for mixed monotone operators $F : X \times X \to X$ obtained in [T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006) 1379-1393] and [N.V. Luong and N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011) 983-992], by weakening the involved contractive condition. An example as well an application to nonlinear Fredholm integral equations are also given in order to illustrate the effectiveness of our generalizations.

1. Introduction and preliminaries

The existence of fixed points and coupled fixed points for contractive type mappings in partially ordered metric spaces has been considered recently by several authors: Ran and Reurings [8], Bhaskar and Lakshmikantham [3], Nieto and Lopez [6], [7], Agarwal et al. [1], Lakshmikantham and Ciric [4], Luong and Thuan [5]. These results found important applications to the study of matrix equations or ordinary differential equations and integral equations, see [8], [3], [6], [7], [5] and references therein.

In order to fix the framework needed to state the main result in [3], we remind the following notions. Let$(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, (u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v.$$

We say that a mapping $F : X \times X \to X$ has the *mixed monotone property* if $F(x, y)$ is monotone nondecreasing in x and is monotone non increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and, respectively,

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

A pair $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping F if

$$F(x, y) = x, F(y, x) = y.$$

The next theorem has been established in [3].

Theorem 1 (Bhaskar and Lakshmikantham [3])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0, 1)$ with*

$$d\left(F\left(x,y\right),F\left(u,v\right)\right)\leq\frac{k}{2}\left[d\left(x,u\right)+d\left(y,v\right)\right],\ \text{for each } x\geq u, y\leq v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F\left(x_0, y_0\right) \ \textit{and}\ y_0 \geq F\left(y_0, x_0\right),$$

then there exist $x, y \in X$ such that

$$x = F\left(x, y\right) \text{ and } y = F\left(y, x\right).$$

As shown in [3], the continuity assumption of F in Theorem 1 can be replaced by the following alternative condition imposed on the ambient space X:

Assumption 1.1. *X has the property that*

(i) if a non-decreasing sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to x, then $x_n \leq x$ for all n;

(ii) if a non-increasing sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to x, then $x_n \geq x$ for all n;

Bhaskar and Lakshmikantham [3] also established uniqueness results for coupled fixed points and fixed points and illustrated these important results by proving the existence and uniqueness of the solution for a periodic boundary value problem. These results were then extended and generalized by several authors in the last five years, see [4], [5] and references therein. Amongst these generalizations, we refer to the ones obtained Luong and Thuan in [5], who have considered instead of (1.1) the more general contractive condition

$$\varphi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq\frac{1}{2}\varphi\left(d\left(x,u\right)+d\left(y,v\right)\right)-\psi\left(d\left(x,u\right)+d\left(y,v\right)\right) \tag{1.2}$$

where φ, $\psi : [0, \infty) \to [0, \infty)$ are functions satisfying some appropriate conditions.

Note that for $\varphi(t) = t$ and $\psi(t) = \frac{1-k}{2}t$, with $0 \leq k < 1$, condition (1.2) reduces to (1.1).

Starting from the results in [3] and [5], our main aim in this paper is to obtain more general coupled fixed point theorems for mixed monotone operators $F : X \times X \to X$ satisfying a contractive condition which is significantly weaker that the corresponding conditions (1.1) and (1.2) in [3] and [5], respectively. We also illustrate how our results can be applied to obtain existence and uniqueness results for integral equations under weaker assumptions than the ones in [5].

2. Main results

Let Φ denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(i_φ) φ is continuous and non-decreasing;

(ii_φ) $\varphi(t) < t$ for all $t > 0$,

and Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy

(i_ψ) $\lim\limits_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim\limits_{t \to 0+} \psi(t) = 0$.

Examples of typical functions φ and ψ are given in [5], see also [2] and [9].

The first main result in this paper is the following coupled fixed point theorem which generalizes Theorem 2.1 in [5] and Theorem 2.1 in [3].

Theorem 2. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ with $x \geq u, y \leq v$,*

$$\varphi\left(\frac{d\left(F\left(x,y\right), F\left(u,v\right)\right) + d\left(F\left(y,x\right), F\left(v,u\right)\right)}{2}\right) \leq$$

$$\leq \varphi\left(\frac{d\left(x,u\right) + d\left(y,v\right)}{2}\right) - \psi\left(\frac{d\left(x,u\right) + d\left(y,v\right)}{2}\right). \tag{2.1}$$

Suppose either

(a) F is continuous or

(b) X satisfy Assumption 1.1.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F\left(x_0, y_0\right) \text{ and } y_0 \geq F\left(y_0, x_0\right), \tag{2.2}$$

or

$$x_0 \geq F\left(x_0, y_0\right) \text{ and } y_0 \leq F\left(y_0, x_0\right), \tag{2.3}$$

then there exist $\overline{x}, \overline{y} \in X$ such that

$$\overline{x} = F\left(\overline{x}, \overline{y}\right) \text{ and } \overline{y} = F\left(\overline{y}, \overline{x}\right).$$

Proof. Consider the functional $d_2 : X^2 \times X^2 \to \mathbb{R}_+$ defined by

$$d_2(Y, V) = \frac{1}{2}\left[d(x,u) + d(y,v)\right], \forall Y = (x,y), V = (u,v) \in X^2.$$

It is a simple task to check that d_2 is a metric on X^2 and, moreover, that, if (X, d) is complete, then (X^2, d_2) is a complete metric space, too. Now consider the operator $T : X^2 \to X^2$ defined by

$$T(Y) = \left(F(x,y), F(y,x)\right), \forall Y = (x,y) \in X^2.$$

Clearly, for $Y = (x, y)$, $V = (u, v) \in X^2$, in view of the definition of d_2, we have

$$d_2(T(Y), T(V)) = \frac{d\left(F\left(x,y\right), F\left(u,v\right)\right) + d\left(F\left(y,x\right), F\left(v,u\right)\right)}{2}$$

and

$$d_2(Y,V)=\frac{d\,(x,u)+d\,(y,v)}{2}.$$

Thus, by the contractive condition (2.1) we obtain that F satisfies the following (φ,ψ)-contractive condition:

$$\varphi\left(d_2(T(Y),T(V))\right)\leq\varphi\left(d_2(Y,V)\right)-\psi\left(d_2(Y,V)\right),\ \forall Y\geq V\in X^2. \tag{2.4}$$

Assume (2.2) holds (the case (2.3) is similar). Then, there exists $x_0,y_0\in X$ such that

$$x_0\leq F(x_0,y_0)\text{ and }y_0\geq F(y_0,x_0).$$

Denote $Z_0=(x_0,y_0)\in X^2$ and consider the Picard iteration associated to T and to the initial approximation Z_0, that is, the sequence $\{Z_n\}\subset X^2$ defined by

$$Z_{n+1}=T(Z_n),\ n\geq 0, \tag{2.5}$$

with $Z_n=(x_n,y_n)\in X^2$, $n\geq 0$.

Since F is mixed monotone, we have

$$Z_0=(x_0,y_0)\leq(F(x_0,y_0),F(y_0,x_0))=(x_1,y_1)=Z_1$$

and, by induction,

$$Z_n=(x_n,y_n)\leq(F(x_n,y_n),F(y_n,x_n))=(x_{n+1},y_{n+1})=Z_{n+1},$$

which shows that the mapping T is monotone and the sequence $\{Z_n\}_{n=0}^{\infty}$ is non-decreasing. Take $Y=Z_n\geq Z_{n-1}=V$ in (2.4) and obtain

$$\varphi\left(d_2(T(Z_n),T(Z_{n-1})\right)\leq\varphi\left(d_2(Z_n,Z_{n-1})\right)-\psi\left(d_2(Z_n,Z_{n-1})\right),\ n\geq 1, \tag{2.6}$$

which, in view of the fact that $\psi\geq 0$, yields

$$\varphi\left(d_2(Z_{n+1},Z_n)\right)\leq\varphi\left(d_2(Z_n,Z_{n-1})\right),\ n\geq 1,$$

which, in turn, by condition (i_φ) implies

$$d_2(Z_{n+1},Z_n)\leq d_2(Z_n,Z_{n-1}),\ n\geq 1, \tag{2.7}$$

and this shows that the sequence $\{\delta_n\}_{n=0}^{\infty}$ given by

$$\delta_n=d_2(Z_n,Z_{n-1}),\ n\geq 1,$$

is non-increasing. Therefore, there exists some $\delta\geq 0$ such that

$$\lim_{n\to\infty}\delta_n=\frac{1}{2}\lim_{n\to\infty}\left[d(x_{n+1},x_n)+d(y_{n+1},y_n)\right]=\delta. \tag{2.8}$$

We shall prove that $\delta=0$. Assume the contrary, that is, $\delta>0$. Then by letting $n\to\infty$ in (2.6) we have

$$\varphi(\delta)=\lim_{n\to\infty}\varphi(\delta_{n+1})\leq\lim_{n\to\infty}\varphi(\delta_n)-\lim_{n\to\infty}\psi(\delta_n)=$$

$$=\varphi(\delta)-\lim_{\delta_n\to\delta+}\psi(\delta_n)<\varphi(\delta),$$

a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} \delta_n = \frac{1}{2} \lim_{n\to\infty} [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = 0. \tag{2.9}$$

We now prove that $\{Z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X^2, d_2), that is, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are Cauchy sequences in (X, d). Suppose, to the contrary, that at least one of the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}_{n=0}^{\infty}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}_{n=0}^{\infty}$ with $n(k) > m(k) \geq k$ such that

$$\frac{1}{2}\left[d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\right] \geq \epsilon,\ k = 1, 2, \ldots. \tag{2.10}$$

Note that we can choose $n(k)$ to be the smallest integer with property $n(k) > m(k) \geq k$ and satisfying (2.10). Then

$$d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) < \epsilon. \tag{2.11}$$

By (2.10) and (2.11) and the triangle inequality we have

$$\epsilon \leq r_k := \frac{1}{2}\left[d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\right] \leq$$

$$\frac{d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1})}{2} + \frac{d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)})}{2}$$

$$\leq \frac{d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1})}{2} + \epsilon.$$

Letting $k \to \infty$ in the above inequality and using (2.9) we get

$$\lim_{k\to\infty} r_k := \lim_{k\to\infty} \frac{1}{2}\left[d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\right] = \epsilon. \tag{2.12}$$

Since $n(k) > m(k)$, we have $x_{n(k)} \geq x_{m(k)}$ and $y_{n(k)} \leq y_{m(k)}$ and hence by (2.1)

$$\varphi(r_{k+1}) = \varphi\Bigg(\frac{1}{2}\left[d\left(F\left(x_{n(k)}, y_{n(k)}\right), F\left(x_{m(k)}, y_{m(k)}\right)\right) +\right.$$

$$\left.\left.+ d\left(F\left(y_{m(k)}, x_{m(k)}\right), F\left(y_{n(k)}, x_{n(k)}\right)\right)\right]\right) \leq \varphi(r_k) - \psi(r_k).$$

Letting $k \to \infty$ in the above inequality and using (2.12) we get

$$\varphi(\epsilon) = \varphi(\epsilon) - \lim_{k\to\infty} \psi(r_k) = \varphi(\epsilon) - \lim_{r_k\to\epsilon+} \psi(r_k) < \varphi(\epsilon),$$

a contradiction. This shows that $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are indeed Cauchy sequences in the complete metric space (X, d).

This implies there exist $\overline{x}, \overline{y}$ in X such that

$$\overline{x} = \lim_{n\to\infty} x_n \text{ and } \overline{y} = \lim_{n\to\infty} y_n.$$

Now suppose that assumption (a) holds. Then

$$\overline{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\overline{x}, \overline{y})$$

and

$$\overline{y}=\lim_{n\to\infty} y_{n+1}=\lim_{n\to\infty} F(y_n,x_n)=F\left(\overline{y},\overline{x}\right),$$

which shows that $(\overline{x},\overline{y})$ is a coupled fixed point of F.

Suppose now assumption (b) holds. Since $\{x_n\}_{n=0}^{\infty}$ is a non-decreasing sequence that converges to $\overline{x}$, we have that $x_n\leq\overline{x}$ for all n. Similarly, $y_n\geq\overline{y}$ for all n.

Then

$$d(\overline{x},F(\overline{x},\overline{y}))\leq d(\overline{x},x_{n+1})+d(x_{n+1},F(\overline{x},\overline{y}))=d(\overline{x},x_{n+1})+$$
$$+d(F(x_n,y_n),F(\overline{x},\overline{y}))$$

and

$$d(\overline{y},F(\overline{y},\overline{x}))\leq d(\overline{y},y_{n+1})+d(y_{n+1},F(\overline{y},\overline{x}))=d(\overline{y},y_{n+1})+$$
$$+d(F(y_n,x_n),F(\overline{y},\overline{x})).$$

So

$$d(\overline{x},F(\overline{x},\overline{y}))-d(\overline{x},x_{n+1})\leq d(F(x_n,y_n),F(\overline{x},\overline{y}))$$

and

$$d(\overline{y},F(\overline{y},\overline{x}))-d(\overline{y},y_{n+1})\leq d(F(y_n,x_n),F(\overline{y},\overline{x}))$$

and hence

$$\frac{1}{2}\left[d(\overline{x},F(\overline{x},\overline{y}))-d(\overline{x},x_{n+1})+d(\overline{y},F(\overline{y},\overline{x}))-d(\overline{y},y_{n+1})\right]\leq$$
$$\leq\frac{1}{2}\left[d(F(x_n,y_n),F(\overline{x},\overline{y}))+d(F(y_n,x_n),F(\overline{y},\overline{x}))\right]$$

which imply, by the monotonicity of φ and condition (2.1),

$$\varphi\left(\frac{1}{2}\left[d(\overline{x},F(\overline{x},\overline{y}))-d(\overline{x},x_{n+1})+d(\overline{y},F(\overline{y},\overline{x}))-d(\overline{y},y_{n+1})\right]\right)\leq$$
$$\leq\varphi\left(\frac{1}{2}\left[d(F(x_n,y_n),F(\overline{x},\overline{y}))+d(F(y_n,x_n),F(\overline{y},\overline{x}))\right]\right)\leq$$
$$\leq\varphi\left(\frac{d(x_n,\overline{x})+d(y_n,\overline{y})}{2}\right)-\psi\left(\frac{d(x_n,\overline{x})+d(y_n,\overline{y})}{2}\right).$$

Letting now $n\to\infty$ in the above inequality, we obtain

$$\varphi\left(\frac{d(\overline{x},F(\overline{x},\overline{y}))+d(\overline{y},F(\overline{y},\overline{x}))}{2}\right)\leq\varphi\left(0\right)-\psi\left(0\right)=0,$$

which shows, by (ii_{φ}), that $d(\overline{x},F(\overline{x},\overline{y}))=0$ and $d(\overline{y},F(\overline{y},\overline{x}))=0$. □

Remark 1. Theorem 2 is more general than Theorem 2.1 in [5] and Theorem 1 (i.e., Theorem 2.1 in [3]), since the contractive condition (2.1) is more general than (1.1) and (1.2), a fact which is clearly illustrated by the next example.

Example 1. Let $X = \mathbb{R}$, $d(x,y) = |x-y|$ and $F : X \times X \to X$ be defined by

$$F(x,y) = \frac{x-2y}{4},\ (x,y) \in X^2.$$

Then F is mixed monotone and satisfies condition (2.1) but does not satisfy neither condition (1.2) nor (1.1).

Indeed, assume there exist $\varphi \in \Phi$ and $\psi \in \Psi$, such that (1.2) holds. This means that for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$,

$$\left| \frac{x-2y}{4} - \frac{u-2v}{4} \right| \leq \frac{1}{2}\varphi\left(|x-u| + |y-v|\right) - \psi\left(|x-u| + |y-v|\right),$$

which, in view of (ii_φ) yields, for $x = u$ and $y < v$,

$$\frac{1}{2}|y-v| \leq \frac{1}{2}\varphi\left(|y-v|\right) - \psi\left(|y-v|\right) \leq \frac{1}{2}\varphi\left(|y-v|\right) < \frac{1}{2}|y-v|,$$

a contradiction. Hence F does not satisfy (1.2).

Now we prove that (2.1) holds. Indeed, since we have

$$\left| \frac{x-2y}{4} - \frac{u-2v}{4} \right| \leq \frac{1}{4}|x-u| + \frac{1}{2}|y-v|,\ x \geq u,\ y \leq v,$$

and

$$\left| \frac{y-2x}{4} - \frac{v-2u}{4} \right| \leq \frac{1}{4}|y-v| + \frac{1}{2}|x-u|,\ x \geq u,\ y \leq v,$$

by summing up the two inequalities above we get exactly (2.1) with $\varphi(t) = t$ and $\psi(t) = \frac{1}{4}t$. Note also that $x_0 = -2$, $y_0 = 3$ satisfy (2.2).

So by our Theorem 2 we obtain that F has a (unique) coupled fixed point $(0,0)$ but neither Theorem 2.1 in [5] nor Theorem 2.1 in [3] do not apply to F in this example.

Remark 2. Note also that Theorem 2.1 in [5] has been proved under the additional very sharp condition on φ:

$$(iii_\varphi)\ \varphi(s+t) \leq \varphi(s) + \varphi(t),\ \forall s,t \in [0,\infty),$$

while our proof is independent of this assumption.

Corollary 1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exists $\psi \in \Psi$ such that for all $x, y, u, v \in X$ with $x \geq u, y \leq v$,*

$$d\left(F(x,y), F(u,v)\right) + d\left(F(y,x), F(v,u)\right) \leq$$
$$\leq d(x,u) + d(y,v) - 2\psi\left(\frac{d(x,u) + d(y,v)}{2}\right). \tag{2.13}$$

Suppose either

(a) F is continuous or

(b) X satisfy Assumption 1.1.

If there exist $x_0, y_0 \in X$ *such that either* (2.2) *or* (2.3) *are satisfied, then there exist* $\overline{x}, \overline{y} \in X$ *such that*

$$\overline{x} = F(\overline{x}, \overline{y}) \text{ and } \overline{y} = F(\overline{y}, \overline{x}).$$

Proof. Taking $\varphi(t) = t$, $t \in [0, \infty)$, condition (2.1) reduces to (2.13) and hence by Theorem 2 we get Corollary 1. □

Remark 3. If we take $\psi(t) = \left(1 - \frac{k}{2}\right) t$, $t \in [0, \infty)$, with $0 \leq k < 1$, by Corollary 1 we obtain a generalization of Theorem 1 (Theorem 2.1 in [3]).

Remark 4. Let us note that, as suggested by Example 1, since the contractivity condition (2.1) is valid only for comparable elements in X^2, Theorem 2 cannot guarantee in general the uniqueness of the coupled fixed point.

It is therefore our interest now to provide additional conditions to ensure that the coupled fixed point in Theorem 2 is in fact unique. Such a condition is the one used in Theorem 2.2 of Bhaskar and Lakshmikantham [3] or in Theorem 2.4 of Luong and Thuan [5]:

every pair of elements in X^2 has either a lower bound or an upper bound, which is known, see [3], to be equivalent to the following condition: for all $Y = (x, y)$, $\overline{Y} = (\overline{x}, \overline{y}) \in X^2$,

$$\exists Z = (z_1, z_2) \in X^2 \text{ that is comparable to } Y \text{ and } \overline{Y}. \tag{2.14}$$

Theorem 3. *In addition to the hypotheses of Theorem 2, suppose that condition* (2.14) *holds. Then* F *has a unique coupled fixed point.*

Proof. From Theorem 2, the set of coupled fixed points of F is nonempty. Assume that $Z^* = (x^*, y^*) \in X^2$ and $\overline{Z} = (\overline{x}, \overline{y})$ are two coupled fixed point of F. We shall prove that $Z^* = \overline{Z}$.

By assumption (2.14), there exists $(u, v) \in X^2$ that is comparable to (x^*, y^*) and $(\overline{x}, \overline{y})$. We define the sequences $\{u_n\}$, $\{v_n\}$ as follows:

$$u_0 = u,\ v_0 = v,\ u_{n+1} = F(u_n, v_n),\ v_{n+1} = F(v_n, u_n),\ n \geq 0.$$

Since (u, v) is comparable to $(\overline{x}, \overline{y})$, we may assume $(\overline{x}, \overline{y}) \geq (u, v) = (u_0, v_0)$. By the proof of Theorem 2 we obtain inductively

$$(\overline{x}, \overline{y}) \geq (u_n, v_n),\ n \geq 0 \tag{2.15}$$

and therefore, by (2.1),

$$\varphi\left(\frac{d(\overline{x}, u_{n+1}) + d(\overline{y}, v_{n+1})}{2}\right) =$$

$$= \varphi\left(\frac{d(F(\overline{x}, \overline{y}), F(u_n, v_n)) + d(F(\overline{y}, \overline{x}), F(v_n, u_n))}{2}\right)$$

$$\leq \varphi\left(\frac{d(\overline{x}, u_n) + d(\overline{y}, v_n)}{2}\right) - \psi\left(\frac{d(\overline{x}, u_n) + d(\overline{y}, v_n)}{2}\right), \tag{2.16}$$

which, by the fact that $\psi \geq 0$, implies

$$\varphi\left(\frac{d(\overline{x}, u_{n+1}) + d(\overline{y}, v_{n+1})}{2}\right) \leq \varphi\left(\frac{d(\overline{x}, u_n) + d(\overline{y}, v_n)}{2}\right).$$

Thus, by the monotonicity of φ, we obtain that the sequence $\{\Delta_n\}$ defined by

$$\Delta_n = \frac{d(\overline{x}, u_n) + d(\overline{y}, v_n)}{2},\ n \geq 0,$$

is non-increasing. Hence, there exists $\alpha \geq 0$ such that $\lim\limits_{n\to\infty} \Delta_n = \alpha$.

We shall prove that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Letting $n \to \infty$ in (2.16), we get

$$\varphi(\alpha) \leq \varphi(\alpha) - \lim_{n\to\infty} \psi(\Delta_n) = \varphi(\alpha) - \lim_{\Delta_n\to\alpha+} \psi(\Delta_n) < \varphi(\alpha).$$

a contradiction. Thus $\alpha = 0$, that is,

$$\lim_{n\to\infty} \frac{d(\overline{x}, u_n) + d(\overline{y}, v_n)}{2} = 0,$$

which implies

$$\lim_{n\to\infty} d(\overline{x}, u_n) = \lim_{n\to\infty} d(\overline{y}, v_n) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(x^*, u_n) = \lim_{n\to\infty} d(y^*, v_n) = 0,$$

and hence $\overline{x} = x^*$ and $\overline{y} = y^*$. $\square$

Corollary 2. *In addition to the hypotheses of Corollary 1, suppose that condition* (2.14) *holds. Then F has a unique coupled fixed point.*

An alternative uniqueness condition is given in the next theorem.

Theorem 4. *In addition to the hypotheses of Theorem 2, suppose that $x_0, y_0 \in X$ are comparable. Then F has a unique fixed point, that is, there exists $\overline{x}$ such that $F(\overline{x}, \overline{x}) = \overline{x}$.*

Proof. Assume we are in the case (2.2), that is

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \leq F(y_0, x_0).$$

Since x_0, y_0 are comparable, we have $x_0 \leq y_0$ or $x_0 \geq y_0$. Suppose we are in the second case. Then, by the mixed monotone property of F, we have

$$x_1 = F(x_0, y_0) \leq F(y_0, x_0) = y_1,$$

and, hence, by induction one obtains

$$x_n \geq y_n,\ n \geq 0. \tag{2.17}$$

Now, since

$$\overline{x} = \lim_{n\to\infty} F(x_n, y_n) \text{ and } \overline{y} = \lim_{n\to\infty} F(y_n, x_n),$$

by the continuity of the distance d, one has

$$d(\overline{x}, \overline{y}) = d(\lim_{n\to\infty} F(x_n, y_n), \lim_{n\to\infty} F(y_n, x_n)) = \lim_{n\to\infty} d(F(x_n, y_n), F(y_n, x_n))$$
$$= \lim_{n\to\infty} d(x_{n+1}, y_{n+1}).$$

On the other hand, by taking $Y = (x_n, y_n)$, $V = (y_n, x_n)$ in (2.1) we have

$$\varphi(d(F(x_n, y_n), F(y_n, x_n))) \leq \varphi(d(x_n, y_n)) - \psi(d(x_n, y_n)),\ n \geq 0,$$

which actually means

$$\varphi(d(x_{n+1}, y_{n+1})) \leq \varphi(d(x_n, y_n)) - \psi(d(x_n, y_n)),\ n \geq 0.$$

Suppose $\overline{x} \neq \overline{y}$, that is $d(\overline{x}, \overline{y}) > 0$. Taking the limit as $n \to \infty$ in the previous inequality, we get

$$\varphi(d(\overline{x}, \overline{y})) = \lim_{n\to\infty} \varphi(d(x_{n+1}, y_{n+1})) \leq \varphi(d(\overline{x}, \overline{y})) - \lim_{n\to\infty} \psi(d(x_n, y_n)),$$

or

$$\lim_{d(x_n, y_n)\to d(\overline{x}, \overline{y})} \psi(d(x_n, y_n)) \leq 0,$$

which contradicts (i_ψ). Thus $\overline{x} = \overline{y}$. □

Remark 5. Note that in [3] and [5] the authors use only condition (2.2), although the alternative assumption (2.3) is also acceptable.

3. Application to integral equations

As an application of the (coupled) fixed point theorems established in Section 2 of our paper, we study the existence and uniqueness of the solution to a Fredholm nonlinear integral equation.

In order to compare our results to the ones in [5], we shall consider the same integral equation, that is,

$$x(t) = \int_a^b (K_1(t, s) + K_2(t, s))\,(f(s, x(s)) + g(s, x(s)))\,ds + h(t), \quad (3.1)$$

$t \in I = [a, b]$.

Let Θ denote the set of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying

(i_θ) θ is non-decreasing;

(ii_θ) There exists $\psi \in \Psi$ such that $\theta(r) = \frac{r}{2} - \psi\left(\frac{r}{2}\right)$, for all $r \in [0, \infty)$.

As shown in [5], Θ is nonempty, as $\theta_1(r) = kr$ with $0 \leq 2k < 1$; $\theta_2(r) = \frac{r^2}{2(r+1)}$; and $\theta_3(r) = \frac{r}{2} - \frac{\ln(r+1)}{2}$, are all elements of Θ.

Like in [5], we assume that the functions K_1, K_2, f, g fulfill the following conditions:

Assumption 3.1. (i) *$K_1(t, s) \geq 0$ and $K_2(t, s) \leq 0$, for all $t, s \in I$;*

(ii) *There exist the positive numbers λ, μ, such that for all $x, y \in \mathbb{R}$, with $x \geq y$, the following Lipschitzian type conditions hold:*

$$0 \leq f(t, x) - f(t, y) \leq \lambda\theta(x - y) \quad (3.2)$$

and

$$-\mu\theta(x-y) \leq g(t,x) - g(t,y) \leq 0; \tag{3.3}$$

(iii)

$$(\lambda+\mu) \cdot \sup_{t\in I} \int_a^b \left[K_1(t,s) - K_2(t,s)\right] ds \leq 1. \tag{3.4}$$

Definition 1. ([5]) A pair $(\alpha,\beta) \in X^2$ with $X = C(I,\mathbb{R})$ is called a *coupled lower-upper solution* of equation (3.1) if, for all $t \in I$,

$$\alpha(t) \leq \int_a^b K_1(t,s)\left[f(s,\alpha(s)) + g(s,\beta(s))\right] ds +$$

$$+ \int_a^b K_2(t,s)\left[f(s,\beta(s)) + g(s,\alpha(s))\right] ds + h(t)$$

and

$$\beta(t) \geq \int_a^b K_1(t,s)\left[f(s,\beta(s)) + g(s,\alpha(s))\right] ds +$$

$$+ \int_a^b K_2(t,s)\left[f(s,\alpha(s)) + g(s,\beta(s))\right] ds + h(t),$$

Theorem 5. *Consider the integral equation* (3.1) *with*

$$K_1, K_2 \in C(I\times I,\mathbb{R}) \text{ and } h \in C(I,\mathbb{R}).$$

Suppose that there exists a coupled lower-upper solution of (3.1) *and that Assumption 3.1 is satisfied. Then the integral equation* (3.1) *has a unique solution in $C(I,\mathbb{R})$.*

Proof. Consider on $X = C(I,\mathbb{R})$ the natural partial order relation, that is, for $x, y \in C(I,\mathbb{R})$

$$x \leq y \Leftrightarrow x(t) \leq y(t),\ \forall t \in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,y) = \sup_{t\in I} |x(t) - y(t)|\,,\ x, y \in C(I,\mathbb{R}).$$

Now define on X^2 the following partial order: for $(x,y), (u,v) \in X^2$,

$$(x,y) \leq (u,v) \Leftrightarrow x(t) \leq u(t), \text{ and } y(t) \geq v(t)\, \forall t \in I.$$

Obviously, for any $(x,y) \in X^2$, the functions $\max\{x,y\}$, $\min\{x,y\}$ are the upper and lower bounds of x, y, respectively.

Therefore, for every $(x,y), (u,v) \in X^2$, there exists the element $(\max\{x,y\}, \min\{x,y\})$ which is comparable to (x,y) and (u,v).

Define now the mapping $F : X \times X \to X$ by

$$F(x,y)(t) = \int_a^b K_1(t,s)\left[f(s,x(s)) + g(s,y(s))\right] ds +$$

$$+ \int_a^b K_2(t,s)\left[f(s,y(s)) + g(s,x(s))\right] ds + h(t), \text{ for all } t \in I.$$

It is not difficult to prove, like in [5], that F has the mixed monotone property. Now for $x,y,u,v\in X$ with $x\geq u$ and $y\leq v$, we have

$$d(F(x,y),F(u,v))=\sup_{t\in I}|F(x,y)(t)-F(u,v)(t)|=$$

$$=\sup_{t\in I}\left|\int_a^b K_1(t,s)\left[f(s,x(s))+g(s,y(s))\right]ds+\right.$$

$$+\int_a^b K_2(t,s)\left[f(s,y(s))+g(s,x(s))\right]ds-$$

$$-\int_a^b K_1(t,s)\left[f(s,u(s))+g(s,v(s))\right]ds-$$

$$\left.-\int_a^b K_2(t,s)\left[f(s,v(s))+g(s,u(s))\right]ds\right|=$$

$$=\sup_{t\in I}\left|\int_a^b K_1(t,s)\left[f(s,x(s))-f(s,u(s))+g(s,y(s))-g(s,v(s))\right]ds+\right.$$

$$\left.+\int_a^b K_2(t,s)\left[f(s,y(s))-f(s,v(s))+g(s,x(s))-g(s,u(s))\right]ds\right|=$$

$$=\sup_{t\in I}\left|\int_a^b K_1(t,s)\left[(f(s,x(s))-f(s,u(s)))-(g(s,v(s))-g(s,y(s)))\right]ds\right.$$

$$\left.-\int_a^b K_2(t,s)\left[(f(s,v(s))-f(s,y(s)))-(g(s,x(s))-g(s,u(s)))\right]ds\right|\leq$$

$$\leq\sup_{t\in I}\left|\int_a^b K_1(t,s)\left[\lambda\theta(x(s)-u(s))+\mu\theta(v(s)-y(s))\right]ds-\right.$$

$$\left.-\int_a^b K_2(t,s)\left[\lambda\theta(v(s)-y(s))+\mu\theta(x(s)-u(s))\right]ds\right|. \tag{3.5}$$

Since the function θ is non-decreasing and $x\geq u$ and $y\leq v$, we have

$$\theta(x(s)-u(s))\leq\theta(\sup_{t\in I}|x(t)-u(t)|=\theta(d(x,u))$$

and

$$\theta(v(s)-y(s))\leq\theta(\sup_{t\in I}|v(t)-y(t)|=\theta(d(v,y)),$$

hence by (3.5), in view of the fact that $K_2(t,s)\leq 0$, we obtain

$$d(F(x,y),F(u,v))\leq\sup_{t\in I}\left|\int_a^b K_1(t,s)\left[\lambda\theta(d(x,u))+\mu\theta(d(v,y))\right]ds-\right.$$

$$\left.-\int_a^b K_2(t,s)\left[\lambda\theta(d(v,y)+\mu\theta(d(x,u))\right]ds\right|=$$

$$=\left[\lambda\theta(d(x,u))+\mu\theta(d(v,y))\right]\cdot\sup_{t\in I}\left|\int_a^b\left[K_1(t,s)-K_2(t,s)\right]ds\right|=$$

$$=\left[\lambda\theta(d(x,u))+\mu\theta(d(v,y))\right]\cdot\sup_{t\in I}\int_a^b\left[K_1(t,s)-K_2(t,s)\right]ds, \tag{3.6}$$

since $K_2(t,s) \le 0$. Similarly, we obtain

$$d(F(y,x),F(v,u)) \le$$

$$= [\lambda\theta(d(v,y)) + \mu\theta(d(x,u))] \cdot \sup_{t\in I} \int_a^b [K_1(t,s) - K_2(t,s)]\, ds. \quad (3.7)$$

By summing (3.6) and (3.7) we get, by using (3.4),

$$\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2} \le (\lambda+\mu)\cdot$$

$$\cdot \sup_{t\in I} \int_a^b [K_1(t,s) - K_2(t,s)]\, ds \cdot \frac{\theta(d(v,y)) + \theta(d(x,u))}{2} \le$$

$$\le \frac{\theta(d(v,y)) + \theta(d(x,u))}{2}.$$

Now, since θ is non-increasing, we have

$$\theta(d(x,u)) \le \theta(d(x,u) + d(v,y)),\ \theta(d(v,y)) \le \theta(d(x,u) + d(v,y))$$

and so

$$\frac{\theta(d(v,y)) + \theta(d(x,u))}{2} \le \theta(d(x,u) + d(v,y)) =$$

$$= \frac{d(v,y) + d(x,u)}{2} - \psi\left(\frac{d(v,y) + d(x,u)}{2}\right),$$

by the definition of θ. Thus we finally get

$$\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2} \le$$

$$= \frac{d(v,y) + d(x,u)}{2} - \psi\left(\frac{d(v,y) + d(x,u)}{2}\right).$$

which is just the contractive condition (2.13) in Corollary 1.

Now, let $(\alpha,\beta) \in X^2$ be a coupled upper-lower solution of (3.1). Then we have

$$\alpha(t) \le F(\alpha(t),\beta(t)) \text{ and } \beta(t) \ge F[\beta(t),\alpha(t)),$$

for all $t \in I$, which show that all hypotheses of Corollary 1 are satisfied. This proves that F has a unique coupled fixed point $(\overline{x},\overline{y})$ in X^2.

Since $\alpha \le \beta$, by Corollary 2 it follows that $\overline{x} = \overline{y}$, that is

$$\overline{x} = F(\overline{x},\overline{x}),$$

and therefore $\overline{x} \in C(I,\mathbb{R})$ is the unique solution of the integral equation (3.1). □

Remark 6. Note that our Theorem 5 is more general than Theorem 3.3 in [5] since, if $\lambda \ne \mu$, then

$$\lambda + \mu < 2\max\{\lambda,\mu\}.$$

For example, if in Assumption 3.1 we have $\lambda = \frac{1}{6}$, $\mu = \frac{1}{12}$ and $\sup_{t\in I} \int_a^b [K_1(t,s) - K_2(t,s)]\, ds = 4$, then our condition (3.4) holds:

$$(\lambda + \mu) \cdot \sup_{t\in I} \int_a^b [K_1(t,s) - K_2(t,s)]\, ds = \frac{1}{4} \cdot 4 \leq 1,$$

so Theorem 5 can be applied but, since

$$2 \max\{\lambda, \mu\} \cdot \sup_{t\in I} \int_a^b [K_1(t,s) - K_2(t,s)]\, ds = 2 \cdot \frac{1}{6} \cdot 4 = \frac{4}{3} > 1,$$

the corresponding condition (iii) in [5] does not hold and hence Theorem 3.3 in [5] cannot be applied to obtain the existence and uniqueness of the solution of the integral equation (3.1).

Remark 7. As a final conclusion, we note that our results in this paper improve all coupled fixed point theorems in [3]-[5], as well as the fixed point theorems in [1], [6]-[8], by considering a more general (symmetric) contractive condition. Note also that our technique of proof reveals that one can use the dual assumption (2.3) for the initial values x_0, y_0 in Theorem 2.

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